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Electric and magnetic gravitational monopoles I. The equation of motion of poles

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Abstract. In this paper, we propose the existence of a new class of particles that we call H poles. Tidal forces that govern the behaviour of nearby H poles in gravitational fields is given by the dual of the Riemann tensor. Consequently, the equivalence principle is not satisfied by these H poles. A physical property of conformal transformation is shown to exist, by means of which the H poles are mapped into particles of geodesic motion. Einstein's equations for the gravitational field in a vacuum are shown to have a larger symmetry group.

1. Introduction

The equations of the empty space gravitational field in Einstein's theory of general relativity can be shown to be equivalent to the vanishing of the divergence of Weyl's tensor, under suitable initial conditions (Lichnerowicz 1960).

By using such a formulation, one can investigate some additional symmetries of Einstein's theory by looking for properties of the electric ($\mathscr{E}_{\alpha\beta}$) and the magnetic ($\mathscr{H}_{\alpha\beta}$) parts of the Weyl tensor. One then realizes that the equations are invariant under a larger group in which the transformation $\mathscr{E}_{\alpha\beta} \to \mathscr{H}_{\alpha\beta}$, $\mathscr{H}_{\alpha\beta} \to -\mathscr{E}_{\alpha\beta}$ is an interesting particular case. This symmetry is, however, broken when source terms are present in the equations. One is then tempted to restore the symmetry by conveniently modifying the equations for the gravitational field. Such modification must be accompanied by the introduction of a class of particles which have a new type of behaviour under the influence of gravitational forces. We will show that such particles are non-minimally coupled with gravitation. The existence of this class should also be expected by considering the analogous situation in Maxwell's electrodynamic equations, as we will see later.

We will divide our present programme of research and analysis of the above symmetry into two parts. In this paper, we will study the behaviour of the new particles in a given gravitational field. Then, in a subsequent paper, we will give the necessary generalization of Einstein's equations of the gravitational field.

The main fact that guided Einstein, using the equivalence principle, to assume the geodesic motion for particles under the unique influence of gravitational forces was the

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well known relationship between inertial and gravitational mass. A particle, thus minimally coupled with gravitation, will be called an E pole†. This represents a straightforward generalization of Newton's theory and it is indeed easy to prove that we can go from a model of Einstein to a Newtonian model by a well defined limiting procedure.

However, a deeper analysis of motion in a gravitational field may be expected to lead us to new qualitative properties of particles in their interaction with gravitation, which do not have a counterpart in Newtonian theory.

To develop this idea in a specific way, we look for the properties of motion of particles in which the curvature of the space–time can be directly shown. In the geodesic equation of motion, the presence of the gravitational force is given by the metrical connection which is coordinate dependent (locally, it can be made to vanish) and, in this sense, cannot be considered as truly observable. This is no longer true for the evolution of the vector η^α that connects, in a congruence of geodesics, two points of neighbouring curves with the same value of the affine parameter. The connecting vector η^α satisfies the Jacobi equation

$$D^2\eta^\alpha/DS^2 = R^{\alpha\mu}_{\beta\nu} V^\nu V_\mu \eta^\beta \tag{1.1}$$

in which $V^\mu = dX^\mu/dS$ is the tangent vector to the geodesics $X^\mu(s)$. S is an affine parameter and $R^{\alpha\mu}_{\beta\nu}$ is the curvature tensor‡, so we can characterize E poles as those particles that move, under the influence of gravitational forces, on curves such that their connecting vector satisfies the Jacobi equation (1.1). This way of describing the behaviour of particles in a given gravitational field itself suggests that we must look for the generalization of the Jacobi equation in order to introduce a new feature on the motion of particles in curved space. The symmetric properties of the Riemann tensor give a unique way of constructing such an equation. Indeed, let $y^\alpha(s)$ be a congruence of curves on the space–time Riemannian manifold such that their connecting vector Π^α satisfies the equation

$$\frac{D^2\Pi^\alpha}{Ds^2} = f R^{*\alpha\mu}_{\beta\nu} V^\nu V_\mu \Pi^\beta \tag{1.2}$$

in which $R^{*\alpha\mu}_{\beta\nu}$ is the dual of the curvature tensor defined as

$$R^*_{\alpha\sigma\beta\nu} = \tfrac{1}{2}\eta_{\alpha\sigma}^{\mu\epsilon} R_{\mu\epsilon\beta\nu}$$

where

$$\eta^{\alpha\beta\mu\nu} = \sqrt{-g}\,\epsilon^{\alpha\beta\mu\nu}$$

and $\epsilon^{\alpha\beta\mu\nu}$ is the totally anti-symmetric Levi–Civita symbol. f is a constant characteristic of each particle. We will call an H pole any particle that moves on curves $y^\alpha(s)$ such that their connecting vector of the congruence, Π^α, satisfies equation (1.2).

The reason for not having a term analogous to the constant f in equation (1.1) reflects the constancy of the ratio of inertial to gravitational mass—and is indeed the main reason for geometrizing gravitational interaction. The new particles do not follow geodesic lines but, as we will see, curves of forced motion. In other words, H poles are not minimally coupled with gravitation.

† The origin for this terminology will be made clear later on.
‡ In our notation, for an arbitrary vector t^α we define the curvature by means of the non-commutative property of a covariant derivative (here represented by a double bar ||):

$$t_{\alpha\|\beta\|\lambda} - t_{\alpha\|\lambda\|\beta} = R_{\alpha\epsilon\beta\lambda} t^\epsilon$$

In § 2 we review Einstein's equation for the gravitational field in terms of electric and magnetic parts of the Weyl tensor; § 3 deals with the equation of motion of H poles, while § 4 gives the conformal behaviour of our theory and shows the interconnection between trajectories of E poles and H poles. In conclusion, in § 5, an analogy with electrodynamics is sketched out.

2. Electric and magnetic parts of the Weyl tensor

The symmetry properties of the Weyl tensor $C_{\alpha\beta\mu\nu}$ enable us to decompose it in terms of two symmetric, trace-free tensors $\mathscr{E}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$ by simple projections of $C_{\alpha\beta\mu\nu}$ and its dual on an arbitrary curve $z^\alpha(s)$ which is specified by its tangent vector S^μ. We write

$$C^{\mu\nu}_{\alpha\beta} = 2S_{[\alpha}\mathscr{E}^{[\mu}_{\beta]}S^{\nu]} + \delta^{[\mu}_{[\alpha}\mathscr{E}^{\nu]}_{\beta]} - \eta_{\alpha\beta\lambda\sigma}S^\lambda\mathscr{H}^{\sigma[\mu}S^{\nu]} - \eta^{\mu\nu\rho\sigma}S_\rho\mathscr{H}_{\sigma[\alpha}S_{\beta]} \tag{2.1}$$

where

$$\mathscr{E}_{\alpha\beta} = -C_{\alpha\mu\beta\nu}S^\mu S^\nu \tag{2.2}$$

is called the electric part of the Weyl tensor and

$$\mathscr{H}_{\alpha\beta} = C^*_{\alpha\mu\beta\nu}S^\mu S^\nu \tag{2.3}$$

is its magnetic part.

Using these two tensors it is possible to write (Jordan *et al* 1960) a set of equations that are equivalent to Einstein's equations under suitable initial data (Lichnerowicz 1960). This set has the same structural composition as Maxwell's equation for electrodynamics. We write them for the case in which the energy–momentum tensor $T_{\mu\nu}$ represents a perfect fluid of density ρ and pressure p:

$$\mathscr{H}^{\mu\lambda}_{\|\nu}h_{\epsilon\lambda}h^\nu_\mu - 3\mathscr{E}_{\alpha\epsilon}\omega^\alpha - \mathscr{E}^\mu_\alpha\theta_{\mu\beta}S_\lambda\eta^{\lambda\beta\alpha}_\epsilon = (\rho+p)\omega_\epsilon \tag{2.4a}$$

$$\mathscr{E}^{\mu\lambda}_{\|\nu}h_{\epsilon\lambda}h^\nu_\mu + 3\mathscr{H}_{\alpha\epsilon}\omega^\alpha + \mathscr{H}^\mu_\alpha\theta_{\mu\beta}S_\lambda\eta^{\lambda\beta\alpha}_\epsilon = \tfrac{1}{3}\rho_{|\alpha}h^\alpha_\epsilon \tag{2.4b}$$

$$\begin{aligned}\dot{\mathscr{E}}^{\mu\rho}h^\sigma_\mu h^\epsilon_\rho + \tfrac{1}{2}\mathscr{E}^{\alpha\mu\|\nu}h^{(\sigma}_\mu\eta^{\epsilon)}_{\lambda\nu\alpha}S^\lambda + \theta\mathscr{H}^{\epsilon\sigma} - \tfrac{1}{2}\mathscr{H}^{(\epsilon}_\nu\theta^{\sigma)\nu} - \tfrac{1}{2}\mathscr{H}^{(\epsilon}_\nu\omega^{\sigma)\nu}\\ - \eta^{\sigma\nu\rho\zeta}\eta^{\epsilon\lambda\alpha\beta}S_\rho S_\lambda\mathscr{H}_{\zeta\alpha}\theta_{\beta\nu} - S^\alpha\mathscr{E}^{\beta(\sigma}\eta^{\epsilon)}_{\lambda\alpha\beta}S^\lambda = 0\end{aligned} \tag{2.5a}$$

$$\begin{aligned}\dot{\mathscr{H}}^{\mu\rho}h^\sigma_\mu h^\epsilon_\rho - \tfrac{1}{2}\mathscr{H}^{\alpha\mu\|\nu}h^{(\sigma}_\mu\eta^{\epsilon)}_{\lambda\nu\alpha}S^\lambda + \theta\mathscr{E}^{\epsilon\sigma} - \tfrac{1}{2}\mathscr{E}^{(\epsilon}_\nu\theta^{\sigma)\nu} - \tfrac{1}{2}\mathscr{E}^{(\epsilon}_\nu\omega^{\sigma)\nu}\\ - \eta^{\sigma\nu\rho\zeta}\eta^{\epsilon\lambda\alpha\beta}S_\rho S_\lambda\mathscr{E}_{\zeta\alpha}\theta_{\beta\nu} + \dot{S}^\alpha\mathscr{H}^{\beta(\sigma}\eta^{\epsilon)}_{\lambda\alpha\beta}S^\lambda = -\tfrac{1}{4}(\rho+p)\sigma_{\epsilon\sigma}.\end{aligned} \tag{2.5b}$$

In these equations, $h_{\mu\nu}$ is the projector on the plane orthogonal to S^μ, that is

$$h_{\mu\nu} = g_{\mu\nu} - S_\mu S_\nu. \tag{2.6}$$

The dot over $\mathscr{E}_{\alpha\beta}(\mathscr{H}_{\alpha\beta})$ indicates the covariant derivative projected in the direction of S_λ, that is

$$\dot{\mathscr{E}}_{\alpha\beta} = \mathscr{E}_{\alpha\beta\|\lambda}S^\lambda.$$

The shear $(\sigma_{\mu\nu})$ and the rotation $(\omega_{\mu\nu})$ tensors and the expansion (θ) are given by

$$\theta = S^\alpha_{\|\alpha} \tag{2.7a}$$

$$\sigma_{\mu\nu} = \tfrac{1}{2}h^\lambda_{(\mu}h^\epsilon_{\nu)}S_{\lambda\|\epsilon} - \tfrac{1}{3}\theta h_{\mu\nu} \tag{2.7b}$$

$$\omega_{\mu\nu} = \tfrac{1}{2}h^\lambda_{[\mu}h^\epsilon_{\nu]}S_{\lambda\|\epsilon}. \tag{2.7c}$$

$\sigma_{\mu\nu}$ is the trace-free part of the tensor of shear $\theta_{\mu\nu}$:

$$\sigma_{\mu\nu} = \theta_{\mu\nu} - \tfrac{1}{3}\theta h_{\mu\nu}. \tag{2.8}$$

Equations (2.4*a*,*b*) give the divergence of $\mathscr{E}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$; equations (2.5*a*,*b*) give a relation between the curl of $\mathscr{E}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$ with the time-dependent $\dot{\mathscr{H}}_{\alpha\beta}$ and $\dot{\mathscr{E}}_{\alpha\beta}$, respectively in a form analogous to Maxwell's theory.

Let us perform an internal transformation in the above set (2.4) and (2.5) that interchanges the role of $\mathscr{E}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$. We set

$$\mathscr{E}_{\alpha\beta} \to \mathscr{E}'_{\alpha\beta} = \cos\phi\,\mathscr{E}_{\alpha\beta} + \sin\phi\,\mathscr{H}_{\alpha\beta} \tag{2.9a}$$

$$\mathscr{H}_{\alpha\beta} \to \mathscr{H}'_{\alpha\beta} = -\sin\phi\,\mathscr{E}_{\alpha\beta} + \cos\phi\,\mathscr{H}_{\alpha\beta} \tag{2.9b}$$

where ϕ represents an arbitrary constant angle. As a consequence of this map, the Weyl tensor changes accordingly:

$$C_{\alpha\beta\mu\nu} \to C'_{\alpha\beta\mu\nu} = \cos\phi\, C_{\alpha\beta\mu\nu} + \sin\phi\, C^{*}_{\alpha\beta\mu\nu}. \tag{2.10}$$

The remarkable fact which we would like to point out here is that in the absence of matter terms, the set (2.4) and (2.5) remains unaltered under such a map. So we realize that there is a new gauge invariance of Einstein's equations by means of which one can rotate the electric and the magnetic parts of the Weyl tensor†.

We assume this symmetry to be a good one in the general case ($T_{\mu\nu} \neq 0$) and we will modify Einstein's equation of the gravitational field conveniently. We shall discuss this modification elsewhere. Here we shall treat the problem of modification of particle behaviour in gravitational fields that becomes necessary as a consequence of the above gauge invariance.

Before doing this, it seems worthwhile to call the attention of the reader to the property of invariance of the super-energy momentum tensor of Bel under the above duality operation.

Bel's tensor (Bel 1962) is a quadratic function of the Weyl tensor written in the form:

$$2T^{\alpha\beta\mu\nu} = C^{\alpha\rho\mu\sigma}C^{\beta\nu}_{\rho\sigma} + C^{*\alpha\rho\mu\sigma}C^{*\beta\nu}_{\rho\sigma}. \tag{2.11}$$

It is trivial to show that this tensor does not change under the map (2.9). So, unless one breaks this symmetry by introducing some non-invariant terms in the equation for fields $\mathscr{E}_{\alpha\beta}$, $\mathscr{H}_{\alpha\beta}$ the fields are not uniquely defined by giving the energy-momentum tensor (2.11).

3. The equation of motion of gravitational poles

In the absence of sources for the contracted Riemann tensor, the Jacobi equation (1.1) and the modified equation (1.2) can be written in terms of the electric and magnetic tensors $\mathscr{E}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$ defined by (2.2) and (2.3), under the form:

$$D^2\eta^{\alpha}/Ds^2 = \mathscr{E}^{\alpha}_{\mu}\eta^{\mu} \tag{3.1}$$

$$D^2\Pi^{\alpha}/Ds^2 = f\mathscr{H}^{\alpha}_{\mu}\Pi^{\mu}. \tag{3.2}$$

† This has a simple expression in two-spinor formalism (Penrose 1960). The spinorial analogue of the Weyl tensor is the completely symmetric fourth-rank spinor ϕ_{ABCD}; the duality rotation $\phi_{ABCD} \to \phi_{abcd}\, e^{i\phi}$ with ϕ a real constant leaves Bianchi's identity in vacuum, namely $\nabla^{AA'}\phi_{ABCD} = 0$, invariant. The tensor equivalent of this duality rotation is precisely (2.10) which implies (2.9).

The origin of the terms E pole and H pole now becomes transparent: they unambiguously denote particles that couple, through two types of tidal forces, with the electric and magnetic parts of the Weyl tensor, respectively.

The equations of motion for E poles are geodesics and the corresponding equations of motion for H poles are curves of forced motion. The acceleration effect on H poles is a completely new phenomenon that has no equivalence in Newtonian theory. So it is a typical effect of the curvature of space–time. We will give here some properties of H-pole trajectories.

Let $y^\alpha(s)$ be the curve under discussion and consider a real parameter s on it. The equation for $y^\alpha(s)$ will be written as

$$\frac{d^2}{ds^2}y^\alpha(s)+\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}\frac{dy^\mu}{ds}\frac{dy^\nu}{ds}=F^\alpha \tag{3.3}$$

where F^α is the forced motion term which induces the deviation of $y^\alpha(s)$ from the equation of a geodesic. Then we construct a family of curves that generate a congruence $y^\alpha(s, v)$ in which v distinguishes different curves and s is a parameter on each curve. Next we impose equation (3.2) on the connecting vector Π^α (that can be defined as the derivative of $y^\alpha(s, v)$ with respect to the v variable). As a consequence, the force F^α must be a solution of the equation

$$F_{\alpha|\mu}-\begin{Bmatrix}\epsilon\\ \alpha\mu\end{Bmatrix}F_\epsilon=\mathscr{E}_{\alpha\mu}+f\mathscr{H}_{\alpha\mu} \tag{3.4}$$

in which we have identified the vector S^μ of equations (2.2) and (2.3) with the tangent vector to the curve $y^\alpha(s)$.

In order to arrive at this condition on F^α a lot of work is saved if we note that the second absolute derivative of Π^α with respect to the s parameter can be written as

$$\frac{D^2\Pi^\alpha}{Ds^2}=\frac{d^2\Pi^\alpha}{ds^2}+\begin{Bmatrix}\alpha\\ \sigma\lambda\end{Bmatrix}_{|\epsilon}\frac{\partial y^\lambda}{\partial s}\frac{\partial y^\epsilon}{\partial s}\Pi^\sigma+\begin{Bmatrix}\alpha\\ \epsilon\sigma\end{Bmatrix}\begin{Bmatrix}\sigma\\ \mu\lambda\end{Bmatrix}\frac{\partial y^\lambda}{\partial s}\frac{\partial y^\epsilon}{\partial s}\Pi^\mu+\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}\Pi^\mu\frac{\partial^2 y^\lambda}{\partial s^2}$$
$$+2\begin{Bmatrix}\alpha\\ \sigma\lambda\end{Bmatrix}\frac{\partial y^\lambda}{\partial s}\frac{d\Pi^\sigma}{ds} \tag{3.5}$$

where

$$\frac{Dt^\alpha}{Ds}\equiv t^\alpha_{\|\beta}\frac{\partial y^\alpha}{\partial s}=\left(t^\alpha_{|\beta}+\begin{Bmatrix}\alpha\\ \epsilon\beta\end{Bmatrix}t^\epsilon\right)\frac{\partial y^\beta}{\partial s}$$

and

$$\frac{dt^\alpha}{ds}=t^\alpha_{|\beta}\frac{\partial y^\beta}{\partial s}\equiv\frac{\partial t^\alpha}{\partial y^\beta}\frac{\partial y^\beta}{\partial s}.$$

Equation (3.4) for F^α seems, at first sight, to be highly involved. In order to know the motion of H poles, we have to know the force F^α. To obtain F^α we must solve equation (3.4) in which the electric and magnetic parts of the Weyl tensor are obtained by projection onto the direction of motion of the H pole. It seems like a 'bootstrap' situation. Fortunately, due to the symmetric properties of H poles we will show that this is only an apparent situation—we can deal very directly with this new kind of motion. The ultimate reason for this simplification rests on the conformal behaviour of our theory.

Before showing this, F^α must fulfill the following special properties in order to be a solution of expression (3.4).

(i) F^α must be a gradient. Indeed, using the symmetric properties of $\mathscr{E}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$ we have

$$F_{\alpha\|\beta} - F_{\beta\|\alpha} = 0.$$

So, due to the symmetric properties of the metric connection $\{^{\alpha}_{\mu\nu}\}$ this implies $F_\alpha = \phi_{|\alpha}$.

(ii) ϕ is a solution of the wave equation ($\Box\phi = 0$) in the given background metric. This can be easily seen by writing equation (3.4) in terms of the ϕ field and making use of the trace-free property of the Weyl tensor.

(iii) F^α is a constant of motion for H poles. Indeed, the absolute derivative of F^α gives

$$\frac{\mathrm{D}\phi^{|\alpha}}{\mathrm{D}s} = \phi^{|\alpha}_{\|\beta}\frac{\partial y^\beta}{\partial s} = (\mathscr{E}^{\alpha}_{\beta} + f\mathscr{H}^{\alpha}_{\beta})\frac{\partial y^\beta}{\partial s} = 0.$$

The last equality comes from the fact that $\mathscr{E}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$ are orthogonal to $\partial y^\alpha/\partial s$. The above property implies that H poles travel on curves of constant acceleration. It is clear that not all such curves can be a possible trajectory of H poles, but any curve $y^\alpha(s)$ of H poles is a curve of constant acceleration.

Notice that equation (3.4) does not restrict the class of metric admissible for a space–time Riemannian manifold. This can be seen by an inspection of the initial data which we have to impose on ϕ and the degrees of freedom of the Weyl tensor.

4. Conformal relationship between E and H trajectories

Consider a a Riemannian manifold V_4 containing a metric $g_{\mu\nu}(x)$ and a set of non-null geodesics, characterized by a generic tangent vector $u^\alpha(s)$, where s is an affine parameter. Let us then project the Weyl tensor and its dual into the u^α directions, according to expressions (2.2) and (2.3) in order to define its electric and magnetic parts.

An arbitrary conformal mapping of V_4 into V_4 generated by a function ψ will be given by setting

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu}(x) = \mathrm{e}^{2\psi(x)} g_{\mu\nu}(x) \tag{4.1a}$$

$$g^{\mu\nu} \to \tilde{g}^{\mu\nu}(x) = \mathrm{e}^{-2\psi(x)} g_{\mu\nu}(x). \tag{4.1b}$$

As a consequence of this mapping, the quantities $\mathscr{E}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$ and the properties of the congruence generated by u^α change accordingly (see equations 2.7):

$$\begin{aligned}
&u^\alpha \to \tilde{u}^\alpha = \mathrm{e}^{-\psi} u^\alpha \\
&\mathscr{E}_{\alpha\beta} \to \tilde{\mathscr{E}}_{\alpha\beta} = \mathscr{E}_{\alpha\beta} \\
&\mathscr{H}_{\alpha\beta} \to \tilde{\mathscr{H}}_{\alpha\beta} = \mathscr{H}_{\alpha\beta} \\
&\theta \to \tilde{\theta} = \mathrm{e}^{-\psi}\theta - 3(\mathrm{e}^{-\psi})_{|\alpha} u^\alpha \\
&\sigma_{\mu\nu} \to \tilde{\sigma}_{\mu\nu} = \mathrm{e}^{\psi}\sigma_{\mu\nu} \\
&\omega_{\mu\nu} \to \tilde{\omega}_{\mu\nu} = \mathrm{e}^{\psi}\omega_{\mu\nu}
\end{aligned} \tag{4.2}$$

in which we have used the invariance property

$$\eta^{\mu\nu}_{\alpha\beta} \rightarrow \tilde{\eta}^{\mu\nu}_{\alpha\beta} = \eta^{\mu\nu}_{\alpha\beta}. \tag{4.3}$$

We would like to call attention to the fact that it is not possible to change the shear-free and/or the rotation-free properties of a congruence of geodesics by a conformal transformation. This is not the case for the acceleration vector. The geodesics $u^\alpha(s)$ are mapped into accelerated curves $\tilde{u}^\alpha(\tilde{s})$ of equations of motion given by

$$\mathrm{D}\tilde{u}^\alpha/\mathrm{D}\tilde{s} = \tfrac{1}{2}(\mathrm{e}^{-2\psi})_{|\beta} g^{\alpha\beta}. \tag{4.4}$$

From the whole class of functions ψ let us select the set $\{\psi \equiv \Phi\}$ such that Φ obeys the equation

$$\tfrac{1}{2}(\mathrm{e}^{-2\Phi})\|^{\alpha}_{\tilde{\sigma}} = \mathrm{e}^{-2\Phi}(\mathscr{E}^\alpha_\sigma + f\mathscr{H}^\alpha_\sigma) \tag{4.5}$$

in which the tilde over σ on the left-hand side of this expression means that the covariant derivative is taken in the conformally transformed metric $\tilde{g}_{\mu\nu}(x)$. By making this choice of functions, we map the class of geodesics $u^\alpha(s)$ into the class of accelerated curves $\tilde{u}^\alpha(\tilde{s})$ defined by equation (3.3). In other words, we map the trajectories of E poles into trajectories of H poles. We remark that the right-hand side of equation (4.5) can be evaluated without reference to the curves of H poles. This is a simple direct consequence of the transformation properties of the Weyl tensor under a conformal mapping. This shows the way of circumventing the bootstrap situation we seemed to be faced with before.

We would like to emphasize here the deep meaning of the conformal transformation that is revealed by our theory. Indeed, conformal mapping appears as an operation by means of which E poles can be transformed into H poles.

Another beautiful consequence of the properties of this mapping is the possibility of deriving the equation of motion of H poles from a variational principle. Expression (4.4) suggests the way for doing this by extremizing the action $A = \int \mathrm{e}^\Phi \,\mathrm{d}\zeta$ for a convenient choice of the function Φ. Indeed, if we set

$$\delta \int \mathrm{e}^\Phi \,\mathrm{d}\zeta = 0 \tag{4.6}$$

then we obtain equation (3.3) for H poles

$$\frac{\mathrm{d}^2 y^\alpha}{\mathrm{d}s^2} + \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} \frac{\mathrm{d}y^\mu}{\mathrm{d}s}\frac{\mathrm{d}y^\nu}{\mathrm{d}s} = \Phi^{|\alpha}.$$

5. Conclusions

In this paper we have introduced a new class of particles which we have called H poles. The crucial property of H poles rests on the fact that the relative acceleration of nearby particles is dominated by tidal forces associated with the dual of the Riemann tensor (the magnetic part of the Weyl tensor in vacuum). This implies that H poles are not minimally coupled with gravitation and so they do not obey the equivalence principle.

The main reason for introducing such a class of particles comes from the behaviour of Einstein's equation in a vacuum, under a rotation of electric and magnetic parts of the Weyl tensor. Indeed, as we have shown in § 2, the invariance of the equations of gravitation under the gauge group (2.9) shows that the role of $\mathscr{E}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$ can be interchanged. This strongly suggests that the usual coupling of particles (E poles) with gravitation through the electric part $\mathscr{E}_{\alpha\beta}$ must be enlarged.

Then, we obtain the most beautiful result of the present paper, i.e., that trajectories of E poles are mapped into trajectories of H poles by a suitable choice of the conformal function. This also gives a new deep insight on the role of the conformal transformation.

Although our result does not make any appeal to similar behaviour in other field theories, it seems worthwhile to compare the present theory with the behaviour of charged particles in the presence of electromagnetic fields.

A very similar situation occurs in Coulomb's discussion on charged particles in electric and magnetic fields. Indeed, in Dirac's version of Coulomb's ideas (Dirac 1948), point particles in electromagnetic fields can be classified as being of two types: e poles or h poles. The equations of motion they satisfy are, respectively:

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2}=\frac{e}{m}F^{\mu\nu}\frac{\mathrm{d}x_\nu}{\mathrm{d}s}\equiv\frac{e}{m}\mathscr{E}^\mu \tag{5.1a}$$

$$\frac{\mathrm{d}^2y^\mu}{\mathrm{d}s^2}=\frac{g}{m}F^{*\mu\nu}\frac{\mathrm{d}y_\nu}{\mathrm{d}s}\equiv\frac{g}{m}\mathscr{H}^\mu \tag{5.1b}$$

in which $\mathscr{E}^\mu$ and $\mathscr{H}^\mu$ are the electric and magnetic fields defined by means of the projection of the antisymmetric tensor $F^{\mu\nu}$ and its dual along the direction of motion. So, we can see the deep formal similarity between electrodynamics and gravidynamics. Indeed, e poles couple to an electric field and h poles couple to a magnetic field in the same way that E poles and H poles, in a gravitational field, couple to the electric and magnetic parts of the Weyl tensor.

If h poles and/or H poles exist in our universe, then we should be able to detect them—at least in principle. A profound dissimilarity between both kind of poles then appears. The existence of h poles does not introduce any change of principles in Maxwell's theory—and if we find such a class of particles (as some authors very recently claim to have done) then the theory of electrodynamics still remains as it is at the present.

This certainly is not the case for H poles. If these particles reveal their existence in the real world then gravitational theory should be changed substantially.

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References

Bel L 1962 *Colloq. Int. CNRS* (Paris: CNRS) p 119
Dirac P A A 1948 *Phys. Rev.* **74** 817
Jordan P, Ehlers J and Kundt W 1960 *Abh. Math.-Naturw. Kl. Akad. Wiss. Mainz* No 2
Lichnerowicz A 1960 *Ann. Mat. Pura Appl.* **50** 1
Penrose R 1960 *Ann. Phys., NY* **10** 171–201